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WEAK OSCILLATIONS OF A GAS BUBBLE IN A SPHERICAL VOLUME OF COMPRESSIBLE LIQUID†

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The following spherically symmetric problem is considered: a single gas bubble at the centre of a spherical flask filled with a compressible liquid is oscillating in response to forced radial excitation of the flask walls. In the long-wave approximation at low Mach numbers, one obtains a system of differential-difference equations generalizing the Rayleigh–Lamb–Plesseth equation. This system takes into account the compressibility of the liquid and is suitable for describing both free and forced oscillations of the bubble. It includes an ordinary differential equation analogous to the Herring–Flinn–Gilmore equation describing the evolution of the bubble radius, and a delay equation relating the pressure at the flask walls to the variation of the bubble radius. The solutions of this system of differential-difference equations are analysed in the linear approximation and numerical analysis is used to study various modes of weak but non-linear oscillations of the bubble, for different laws governing the variation of the pressure or velocity of the liquid at the flask wall. These solutions are compared with numerical solutions of the complete system of partial differential equations for the radial motion of the compressible liquid around the bubble.

1. FORMULATION OF THE PROBLEM

Mathematical studies of the radial pulsations of a gas bubble in a homogeneous liquid previously took one of two forms. One approach was to assume that the liquid is unbounded and incompressible, but with the pressure at infinity specified. The equation of motion of the liquid was then reduced to the Rayleigh–Lamb–Plesseth equation [1–3]

$$a\frac{dw_a}{dt}+\frac{3}{2}w_a^2=\frac{p_a-p_\infty}{\rho},\quad w_a=\frac{da}{dt},\quad p_a(a)=p_g(a)-\frac{2\Sigma}{a} \tag{1.1}$$

where ρ is the density, Σ is the surface tension of the liquid, w_a is the radial velocity of the liquid at the bubble surface, p_g is the gas pressure in the bubble (more precisely, at its walls), p_a is the liquid pressure at the bubble wall, and p_∞ is the pressure far away from the bubble (at infinity). The other method was to allow for the weak compressibility of the unbounded liquid surrounding the bubble by adopting the so-called scheme of linear or non-linear acoustic radiation. In that case the radial motion of the bubble was described by the Herring–Flinn–Gilmore equation [4]

$$a\frac{dw_a}{dt}+\frac{3}{2}w_a^2=\frac{p_a-p_\infty}{\rho}+\frac{a}{\rho C}\frac{d(p_a-p_\infty)}{dt} \tag{1.2}$$

Allowance was made for the damping of the oscillations owing to spherical acoustic pressure waves radiating from the bubble. The pressure p_∞ was interpreted as the pressure in the liquid far from the bubble, but no method for calculating it was specified.

An approximate theory, based on the perturbation method, was developed for the radial motions of a spherical bubble in an unbounded compressible liquid on the assumption that the liquid is not perturbed at infinity [5]. This theory yielded a "family" of equations for the bubble oscillations, which includes Eq. (1.2) and the equations obtained by others [4] as special cases. It has been shown that all these equations are "equivalent", in the sense that they are accurate to the same order with respect to the Mach number. It was assumed that the bubble oscillations have no influence on the external acoustic pressure field in the liquid, i.e. a wave incident on the bubble is reflected from it unchanged. This enabled

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the problem of the oscillating bubble to be treated in isolation from the acoustic problem in the liquid, using the external acoustic field as a given driving force acting on the bubble.

In this paper we consider the combined problem of the oscillations of the bounded volume of liquid and the gas bubble. It will be shown that the problem may be reduced to an equation similar to the Herring–Flinn–Gilmore equation for the bubble radius, in which the dependence of the driving pressure on the change in the bubble radius and the pressure at the outer boundary of the liquid (the flask wall) is given by a differential-difference equation.

Consider a liquid in a spherical flask of radius R and a spherical bubble of radius a at the centre of the flask. We will study the spherically symmetric radial motions of the liquid around the bubble, due to small spherically symmetric displacements of the flask wall, of amplitude δr, assuming that

$$a \ll R, \quad \delta r \ll R \tag{1.3}$$

One might think that when $a \ll R$ the pressure far from the bubble may be taken equal to the pressure p_R at the flask wall, i.e. $p_\infty = p_R$. However, this is not generally true.

First, because of the finite speed at which the perturbations propagate in the liquid (the finite speed of sound in the liquid), there is a time delay t_d between the perturbation at the flask wall and the arrival of this perturbation in the vicinity and, in particular, at the wall of the bubble. For $R_F = 5$ cm and $C = 1500$ m/s one has $t_d = 33$ μs which is comparable with (or even greater than) the oscillation period if the frequency of the imposed perturbation is $f > 10$ kHz.

Second, a perturbation originating at the outer boundary of the liquid (the flask wall), owing to its spherical acoustic convergence as it penetrates the flask, is amplified as r^{-1}, where r is the radial coordinate measured from the centre of the spherical flask. Hence the bubble will "sense" the amplification of the perturbation compared with its amplitude at the boundary of the flask $r = R$.

Thus, the Rayleigh–Lamb–Plesseth equation (1.1) and the Herring–Flinn–Gilmore equation (1.2), both of which assume that the liquid is infinite, are not suitable for a gas bubble in a finite volume of compressible liquid.

The spherically symmetric formulation of the problem of radial motions (the field of radial velocities $w(r, t)$) of a compressible liquid around a spherical bubble, which includes differential equations for the mass and the momentum, the barotropic state equation of the liquid at pressure p and density ρ, and the boundary conditions at the bubble surface $r = a$ and the flask wall $r = R$, may be written as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho w)}{\partial r} + \frac{2\rho w}{r} = 0, \quad \rho\frac{\partial w}{\partial t} + \rho w \frac{\partial w}{\partial r} + \frac{\partial p}{\partial r} = 0, \quad p = p(\rho) \tag{1.4}$$

$$r = R\colon\ p = p_R(t); \quad r = a\colon\ p = p_a(a) = p_g(a) - \frac{2\Sigma}{a}, \quad w = w_a = \frac{da}{dt} = a' \tag{1.5}$$

2. THE EQUATIONS OF THE RADIAL OSCILLATIONS OF THE BUBBLE

It can be shown that the space between the bubble surface and the outer surface of the flask consists of three zones:

1. A far or external zone, where the compressibility of the liquid is significant but the non-linear inertial forces produced by convective accelerations are negligibly small and the motion of the liquid is wave-like.

2. The neighbourhood of the bubble, or internal zone, where the liquid may be considered to be incompressible and the motion is due only to contraction and expansion of the bubble, but the non-linear inertial forces produced by convective accelerations are significant.

3. An intermediate zone, where the compressibility of the liquid and the non-linear inertial forces produced by convective accelerations are fairly large.

In the first two zones one can construct appropriate asymptotic analytical solutions.

In the external or far zone, far away from the bubble ($r^2 \gg a^2$), the convective accelerations of the liquid particles are fairly small

$$w\partial w / \partial r \ll \partial w / \partial t$$

This estimate follows from estimates for the accelerations

$$w\partial w / \partial r \approx w_R^2 / \lambda_R, \quad \partial w / \partial t \approx w_R / t_R \quad (\lambda_R = Ct_R)$$

where λ_R and t_R are the length and period of the wave perturbation in the liquid in the external zone and t_R is equal to the characteristic period of the vibrations at the flask wall. The ratio of these accelerations is equal to the characteristic Mach number M_R of the motion of the flask wall, which is assumed to be small

$$w\partial w / \partial r / (\partial w / \partial t) \sim w_R t_R / \lambda_R = w_R / C = M_R \ll 1$$

Then the momentum and mass equations of the continuous medium in the external domain ($r \gg a$, $w < w_{ex}, p < p_{ex}$) reduce to the linear wave equation for the external asymptotic behaviour of the velocity potential φ_{ex}, which may be written as follows for spherically symmetric motion

$$\frac{\partial^2 \varphi_{ex}}{\partial t^2} = C^2 \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi_{ex}}{\partial r}\right)$$

The general solution of this equation is

$$\varphi_{ex} = \frac{1}{r}\left[\psi_1\left(t - \frac{r}{C}\right) + \psi_2\left(t + \frac{r}{C}\right)\right]$$

where ψ_2 characterizes a wave incident on the bubble and ψ_2 characterizes a wave reflected from the bubble.

Using the Cauchy–Lagrange integral for transient potential motion and the smallness of the perturbations in the external zone ($r^2 \gg a^2$), one finds that the external asymptotic behaviour of the pressure p_{ex}, the velocity w_{ex} and the density ρ_{ex} may be described in terms of the asymptotic potential φ_{ex} as follows:

$$p_{ex} = p_0 - \rho_0 \frac{\partial \varphi_{ex}}{\partial t}, \quad w_{ex} = \frac{\partial \varphi_{ex}}{\partial r}, \quad \rho_{ex} = \rho_0 + \frac{p_{ex} - p_0}{C^2} \tag{2.1}$$

In the second zone, or the region next to the bubble ($r \sim (1 \div 10)a$), that is, in the boundary layer, which is thin compared with the radius of the flask, an estimate derived from the equation of conservation of mass (the first equation in (1.4)) yields the following formulae

$$\frac{\partial \rho}{\partial t} \approx \frac{\delta \rho}{t_a}, \quad w\frac{\partial \rho}{\partial r} \approx w_a \frac{\delta \rho}{a}, \quad \rho \frac{\partial w}{\partial r} \approx \rho \frac{w_a}{a}, \quad \frac{\rho w}{r} \approx \frac{\rho w_a}{a} \tag{2.2}$$

where t_a is the characteristic time of a density wave around the bubble and $w_a = a'$ is the characteristic velocity of motion of the liquid at the bubble surface. The assumption that near the bubble the displacements of the liquid (due to the compressibility of the bubble) in the characteristic time t_a are comparable with the bubble radius a may be written as follows:

$$t_a \approx a / w_a$$

At low Mach numbers near the bubble ($M_a = w_a/C \ll 1$), this assumption corresponds to long density waves near the bubble: $\lambda_a \sim Ct_a \gg a$.

In sum: the ratio of the terms associated with the compressibility ($\delta\rho$) to terms not associated with the compressibility is equal to the relative variation of the liquid density around the bubble, which is assumed to be very small

$$\frac{\delta \rho}{t_a} \frac{a}{\rho w_a} \approx \frac{\delta \rho}{\rho} \equiv \varepsilon_p \ll 1, \quad \frac{w_a \delta \rho}{a} \frac{a}{\rho w_a} \approx \frac{\delta \rho}{\rho} \equiv \varepsilon_p \ll 1$$

Therefore, in a region close to the bubble but in the internal zone ($\varphi = \varphi_{in}$), the following asymptotic formula holds for an incompressible liquid

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi_{in}}{\partial r}\right)=0$$

The solution of this equation, taking into account the boundary condition at the bubble wall ($r = a$: $w = w_a = a'$), is

$$\varphi_{in} = -a'a^2 / r \tag{2.3}$$

The Cauchy–Lagrange integral in this case includes a component corresponding to the non-linear inertial force; the integral is

$$\frac{\partial \varphi}{\partial t}+\frac{w^2}{2}+\frac{p}{\rho_0}=F(t) \tag{2.4}$$

In view of (2.3), this integral yields the Rayleigh–Lamb–Plesseth equation (1.1), where p_∞ is the pressure at "internal infinity"

$$r = R_i:\ p = p_\infty\ (a \ll R_i \ll R) \tag{2.5}$$

To obtain an equation for the radial motion of the gas bubble for a given perturbation at the flask wall ($p_R = p_R(t)$), taking into account the compressibility of the liquid, one must "match" the asymptotic solutions for the external and internal zones in the intermediate zone or at intermediate infinity. The matching condition must be used for the volume flow rate of the liquid and for the pressure

$$4\pi r^2\left.\frac{\partial \varphi_{in}}{\partial r}\right|_{r\to\infty}=4\pi r^2\left.\frac{\partial \varphi_{ex}}{\partial r}\right|_{r\to 0},\quad p_{in}|_{r\to\infty}=p_{ex}|_{r\to 0} \tag{2.6}$$

Taking into account the fact that the flow rate for an incompressible liquid depends only on time

$$Q(t)=r^2\left.\frac{\partial \varphi_{in}}{\partial r}\right|_{r\to\infty}\equiv a^2\frac{da}{dt}$$

one can express the first relationship in (2.6) in the form

$$Q(t)=r^2\left.\frac{\partial \varphi_{ex}}{\partial r}\right|_{r\to\infty}=-\left[\psi_1\left(t-\frac{r}{C}\right)+\psi_2\left(t+\frac{r}{C}\right)\right]+$$

$$+\frac{r}{C}\left[-\psi_1'\left(t-\frac{r}{C}\right)+\psi_2'\left(t+\frac{r}{C}\right)\right]_{r\to 0}\to -\psi_1(t)-\psi_2(t)$$

implying the following relationship between the reflected and incident waves

$$\psi_1=-\psi_2-Q \tag{2.7}$$

and the asymptotic formula for the velocity potential in the far zone is

$$\varphi_{ex}=\frac{1}{r}\left[\psi_2\left(t+\frac{r}{C}\right)-\psi_2\left(t-\frac{r}{C}\right)-Q\left(t-\frac{r}{C}\right)\right] \tag{2.8}$$

Using the Cauchy–Lagrange integrals (the first expression of (2.1) and (2.4)) and the asymptotic formulae for the velocity potentials (2.1) and (2.3), one finds asymptotic formulae for the pressure distributions in the external and internal zones

$$p_{ex}=p_0-\rho\frac{\partial \varphi_{ex}}{\partial t}=p_0-\frac{\rho}{r}\left[\psi_2'\left(t+\frac{r}{C}\right)-\psi_2'\left(t-\frac{r}{C}\right)-Q'\left(t-\frac{r}{C}\right)\right] \tag{2.9}$$

$$p_{in}=p_a(a)-\rho\left(aa''+\frac{3}{2}(a')^2\right)+\frac{\rho}{r}(a^2a')'-\frac{\rho}{2}\frac{a^4(a')^2}{r^4} \tag{2.10}$$

The internal asymptotic behaviour ($r \to 0$) of the external solution p_{ex} in (2.9) may be expressed as follows:

$$p_{ex}\big|_{r\to 0} = p_0 - \lim_{r\to 0}\frac{\rho}{r}\left[\psi_2'\left(t+\frac{r}{C}\right)-\psi_2'\left(t-\frac{r}{C}\right)-Q'\left(t-\frac{r}{C}\right)\right]=$$
$$= p_0 - \frac{2\rho\psi_2''(t)}{C}+\frac{\rho Q'(t)}{r}+\frac{\rho Q''(t)}{C} \quad (Q' \equiv (a^2a')') \tag{2.11}$$

Estimating the limits of the last two asymptotic expressions using the second formula of (2.6), one obtains the long-wave approximation of the equation of radial oscillations in a compressible liquid

$$aa''+\frac{3}{2}a'^2 = \frac{p_a(a)-p_0}{\rho}+\frac{1}{C}[2\psi_2''(t)+Q''(t)] \tag{2.12}$$

It is remarkable that the term $\rho Q'(t)/r \equiv \rho(a^2a')'/r$ is the same both in the expression for p_{ex} and in the expression for p_{in} (compare (2.10) and (2.11)), so that the asymptotic formulae can be compared.

The pressure at the flask wall may be expressed as follows, taking into account the external asymptotic formula (2.9)

$$p_F = p_0 - \frac{\rho}{R}\left[\psi_2'\left(t+\frac{R}{C}\right)-\psi_2'\left(t-\frac{R}{C}\right)-Q'\left(t-\frac{R}{C}\right)\right] \tag{2.13}$$

The term $Q''(t)/C$ represents the effect of the bubble on the reflected wave; it involves the third derivative $a'''(t)$ and is fairly small in the long-wave approximation. Indeed, the components of Eq. (2.12) may be estimated in the same way as (2.2)

$$aa'' \sim (a')^2 \sim \frac{a^2}{t_a^2},\quad \frac{1}{C}Q''(t) \equiv \frac{1}{C}(a^2a')'' \sim \frac{a^3}{t_a^3C} \tag{2.14}$$

and the ratio between them is determined by a small parameter in the internal zone (around the boundary layer of the bubble), given the validity, as assumed, of the long-wave approximation and the smallness of the Mach number

$$\frac{Q''/C}{aa''} \sim \frac{Q''/C}{Q'/a} \sim \frac{a}{t_aC} \sim \frac{a}{\lambda_a} \sim \frac{a'}{C} \ll 1 \quad \left(\frac{a}{\lambda_a} \equiv \varepsilon_a, \frac{a'}{C} \equiv M_a\right) \tag{2.15}$$

Noting that

$$\frac{Q'}{a} \equiv aa''+2a'^2,\quad aa''+\frac{3}{2}a'^2 \equiv \frac{Q'}{a}-\frac{1}{2}a'^2$$

we can rewrite (2.12) as

$$\frac{Q'}{a}-\frac{1}{2}a'^2 = \frac{p_a(a)-p_{ef}}{\rho}+\frac{1}{C}Q''(t),\quad p_{ef} = p_0 - \frac{2\rho}{C}\psi_2'' \tag{2.16}$$

If one assumes, in accordance with (2.15), that the term $Q''(t)/C$ is small, then the first equation of (2.16) can be simplified considerably

$$Q' = a\left[\frac{1}{2}a'^2+\frac{p_a(a)-p_{ef}}{\rho}\right](1+O(\varepsilon_a))$$

Differentiating this equality with respect to time, we obtain a long-wave approximation at low Mach numbers for $Q''(t)/C$

$$\frac{Q''}{C}(1+O(\varepsilon_a)) = \left[aa''+\frac{a'^2}{2}+\frac{p_a(a)-p_{ef}}{\rho}\right]\frac{a'}{C}+\frac{a}{C}\frac{d}{dt}\left[\frac{p_a(a)-p_{ef}}{\rho}\right] \tag{2.17}$$

The components of the right-hand side of this equation may be estimated in the same way as (2.14) as of the first order of smallness compared with the principal terms of Eq. (2.12)

$$\left[aa''+\frac{a'^2}{2}+\frac{p_a(a)-p_{\mathrm{ef}}}{\rho}\right]\frac{a'}{C}\sim\frac{a}{C}\frac{d}{dt}\left[\frac{p_a(a)-p_{\mathrm{ef}}}{\rho}\right]\sim\frac{\Delta p}{\rho}\varepsilon_a$$

and the implicit remainder term is of the second order of smallness compared with the principal terms of Eq. (2.12)

$$\frac{Q''}{C}O(\varepsilon_a)=\frac{\Delta p}{\rho}(\varepsilon_a)^2$$

Ignoring quantities of order ε_a^2 compared with unity (the long-wave approximation and the low Mach number conditions assumed in the boundary layer around the bubble) and substituting formulae (2.17) into (2.12), we have

$$\left(1-\frac{a'}{C}\right)aa''+\frac{3}{2}\left(1-\frac{a'}{3C}\right)a'^2=\left(1+\frac{a'}{C}\right)\frac{p_a(a)-p_{\mathrm{ef}}}{\rho}+\frac{a}{C}\frac{d}{dt}\left[\frac{p_a(a)-p_{\mathrm{ef}}}{\rho}\right]$$

Thus the evolution of the bubble radius in a weakly compressible liquid ($\varepsilon_\rho\equiv\Delta\rho/\rho\ll 1$) due to radial displacements of the flask walls in the long-wave approximation ($\varepsilon_a\equiv a/\lambda_a\ll 1$) at low Mach numbers ($M_a\equiv a'/C\sim\varepsilon_a\ll 1$) is described by the following system of equations

$$aa''+\frac{3}{2}a'^2=\frac{p_a(a)-p_{\mathrm{ef}}}{\rho}+\frac{a}{C}\frac{d}{dt}\left(\frac{p_a(a)-p_{\mathrm{ef}}}{\rho}\right),\quad p_{\mathrm{ef}}=p_o-\frac{2\rho}{C}\psi_2''$$

$$p_R(t)=p_0-\frac{\rho}{R}\left[\psi_2'\left(t+\frac{R}{C}\right)-\psi_2'\left(t-\frac{R}{C}\right)-Q'\left(t-\frac{R}{C}\right)\right]\tag{2.18}$$

This system is closed, given the state equation for the gas in the bubble with surface tension $p_g=p_g(a)$, initial data $t=0$: $a=a_0$, $a'=a_0'$ and the force acting on the flask walls, in particular, the pressure at the flask walls $p_R=p_R(t)$.

Finally, using the simplified expression (2.17) for Q'' in the long-wave approximation at low Mach numbers, one can rewrite the system of equations (2.18) as a system of differential-difference equations with a recurrence reaction for $p_{\mathrm{ef}}(t)$

$$aa''+\frac{3}{2}a'^2=\frac{p_a(a)-p_{\mathrm{ef}}(t)}{\rho_0}+\frac{a}{C}\frac{d}{dt}\left(\frac{p_a(a)-p_{\mathrm{ef}}(t)}{\rho_0}\right)$$

$$p_{\mathrm{ef}}\left(t+\frac{R}{C}\right)=p_{\mathrm{ef}}\left(t-\frac{R}{C}\right)+\frac{2R}{C}p_R'(t)-\frac{2a}{C}\frac{d}{dt}(p_a(a)-p_{\mathrm{ef}})|_{t-R/C}\tag{2.19}$$

$$p_a(a)=p_g(a)-\frac{2\Sigma}{a},\quad p_g(a)=\left(p_0+\frac{2\Sigma}{a_0}\right)\left(\frac{a}{a_0}\right)^{-3\gamma}$$

The system of equations (2.19) was first proposed in [6] to describe the slow (low Mach number) stage of variation of the radius of a gas bubble in experiments on sololuminescence (oscillations of the bubble under the action of a powerful acoustic field, accompanied by luminescence of the gas).

Experimental verification of the proposed model requires simultaneous measurement of both the gas bubble size dynamics and the pressure of the liquid at the flask walls.

It should be noted that, generally speaking, the spherical symmetry of the problem is violated owing to buoyancy forces. Computations show, however, that the upward displacement of the bubble (buoyancy) is quite small compared with the size of the flask (it amounts to a few bubble radii). The problem of the stability of the bubble position at the flask centre was considered in [9].

3. THE AMPLITUDE-FREQUENCY RESPONSE CURVE

To construct the amplitude–frequency response of the radial pulsations of the bubble one must linearize system (2.19) and determine the amplitude of the perturbation of the bubble radius and the effective pressure as functions of the pressure at the flask walls and the frequency, for sinusoidal perturbations of frequency ω.

The linearization of the system of equations for the perturbations

$$\Delta a = a - a_0, \quad \Delta p_{ef} = p_{ef} - p_0, \quad \Delta p_R = p_R - p_0$$

has the form

$$\frac{\Delta a''}{a_0} + \delta\frac{\Delta a'}{a_0} + \omega_0^2\frac{\Delta a}{a_0} = -\frac{p_0}{\rho_0 a_0^2}\left[\frac{\Delta p_{ef}}{p_0} + \frac{\Delta a}{a_0}\frac{\Delta p'_{ef}}{p_0}\right]$$

$$\frac{\Delta p_{ef}(t+R/C)}{p_0} = \frac{\Delta p_{ef}(t-R/C)}{p_0} + \frac{2R}{C}\frac{\Delta p'_R(t)}{p_0} + \tag{3.1}$$

$$+\frac{2\rho_0 a_0^3\omega_0^2}{p_0 C}\frac{\Delta a'(t-R/C)}{a_0} + \frac{2a_0}{C}\frac{\Delta p'_{ef}(t-R/C)}{p_0}$$

$$\omega_0^2 = \frac{p_0}{\rho_0 a_0^2}[3\gamma + (3\gamma-1)\sigma], \quad \delta = \omega_0^2\frac{a_0}{C}, \quad \sigma = \frac{2\Sigma}{p_0 a_0}$$

Assuming sinusoidal perturbations

$$\frac{\Delta p_R(t)}{p_0} = P_R e^{i\omega t}, \quad \frac{\Delta p_{ef}(t)}{p_0} = P_{ef}e^{i\omega t}, \quad \frac{\Delta a(t)}{a_0} = Ae^{i\omega t}$$

one can reduce the linearized system (3.1) to the equations

$$A = -\frac{p_0}{\rho_0 a_0^2}\frac{1+i\omega a_0/C}{\omega_0^2 + i\delta\omega - \omega^2}P_{ef}$$

$$P_{ef}\sin\left(\frac{\omega R}{C}\right) = \frac{\omega R}{C}P_R + \frac{a_0\omega}{C}\exp\left(-\frac{i\omega R}{C}\right)\left(\frac{\rho_0 a_0^2\omega_0^2}{p_0}A + P_{ef}\right) \tag{3.2}$$

Introducing dimensionless variables

$$x = \frac{\omega R}{C}, \quad \varepsilon = \frac{a_0}{R}, \quad \Gamma = 3\gamma + (3\gamma-1)\sigma, \quad E = \frac{\rho_0 C^2}{p_0\Gamma}$$

and substituting the first equation of (3.2) into the second, we obtain expressions for the amplitude ratios of the perturbations

$$\frac{A}{P_{ef}} = -\frac{1}{\Gamma}\frac{1+i\varepsilon x}{1+i\varepsilon x - E\varepsilon^2 x^2}$$

$$\frac{P_{ef}}{P_R} = x\left[\sin x + \exp(-ix)\frac{E\varepsilon^3 x^3}{1+i\varepsilon x - E\varepsilon^2 x^2}\right]^{-1} \tag{3.3}$$

$$\frac{A}{P_R} = A_R(x) =$$

$$= -\frac{x(1+i\varepsilon x)}{\Gamma}[(1+i\varepsilon x - E\varepsilon^2 x^2)\sin x + E\varepsilon^3 x^3\exp(-ix)]^{-1}$$

Hence it follows that the absolute value of A_R is

$$|A_R| = \frac{x(1+\varepsilon^2 x^2)^{1/2}}{\Gamma}\times$$

$$\times\{[E\varepsilon^3x^3\cos x+(1-E\varepsilon^2x^2)\sin x]^2+(1-E\varepsilon^2x^2)^2\varepsilon^2x^2\sin^2 x\}^{-1/2} \tag{3.4}$$

Clearly, as $\varepsilon \to 0$, when the effect of the bubble on the liquid dynamics disappears, the solution of problem (3.3) degenerates into a solution corresponding to a monochromatic standing spherical acoustic wave [7]

$$P_{ef} / P_R = x / \sin x$$

The quantity P_{ef} equals the amplitude of the pressure in the liquid at the centre of the flask. The frequencies $\omega_k = k\pi C/R$ ($x_k = k\pi$) at which the liquid pressure amplitude at the centre tends to infinity correspond to the case of "flask" resonance. This means that the ratio of the propagation time of a wave from the flask wall to the centre and back ($2R/C$) to the oscillation period of the flask walls ($2\pi/\omega_k$) is an integer k.

Typical amplitude–frequency response curves $|A_k|$ (3.4) are shown in Fig. 1 for the case of a spherical flask of radius R = 5 cm filled with water (C = 1500 m/s, $p_0 = 10^5$ Pa, $\rho_0 = 10^3$ kg/m^3, Σ = 0.073 N/m), with a gas bubble (γ = 1.4) of radius a_0 = 10, 100, 500 and 500 μm situated at the flask centre. The first "flask" resonance occurs at frequency ω_1 = 94.2 kHz.

For fine bubbles, whose radius is very small compared with the flask radius ($\varepsilon \ll 1$), resonance (maximum $|A_k|$) occurs when $\sin x' = 0$, or $x = x_k = k\pi$. Thus, the bubble has practically no effect on the resonance frequency. It will influence the resonance frequency only when the oscillation frequency of the flask is comparable with the resonance frequency ω_0 of the bubble (cf. the curves for a_0 = 100 μm).

It is noteworthy that outside the bubble resonance zone, the smaller the bubble, the higher is the value of the response function $|A_k|$ (cf. the corresponding curves for a_0 = 10 and 500 μm).

For bubbles of sufficiently large radius, but still significantly smaller than the flask, the parameter ε may considerably affect the shape of the response curve, both with respect to the value of the resonance frequencies (which may differ substantially from the flask resonance frequencies) and the value of the response function in the resonance domain (the case a_0 = 5000 μm).

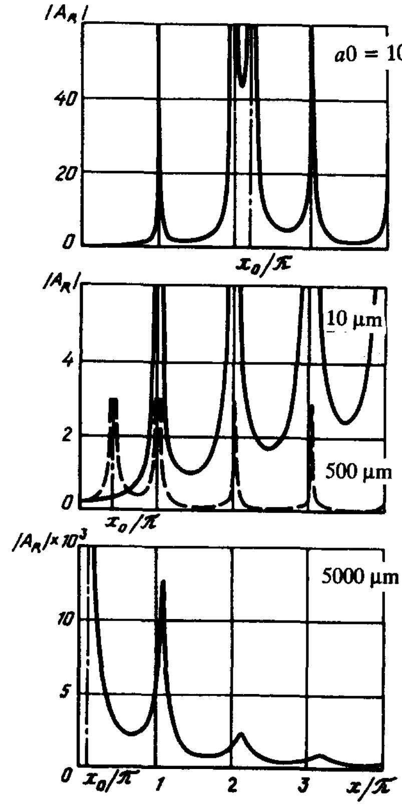

Fig. 1.

4. NUMERICAL INVESTIGATION OF THE COMPLETE SYSTEM OF EQUATIONS

We will now consider problem (1.4), (1.5) of the radial motion of a compressible liquid around a gas bubble at the centre of a flask, on the assumption that the pressure on the flask wall varies according to a given law. In Lagrange variables (r_0, t), system (1.4) takes the following form

$$\frac{\rho_0}{\rho}-\left(\frac{r}{r_0}\right)^2\frac{\partial r}{\partial r_0}=0,\ \rho_0\frac{\partial w}{\partial t}+\left(\frac{r}{r_0}\right)^2\frac{\partial p}{\partial r_0}=0,\ \frac{\partial r}{\partial t}=w,\ p=p(\rho) \tag{4.1}$$

The boundary and initial data are

$$\begin{gathered}p(R,t)=p_R(t),\ p(a_0,t)=p_g(a)-2\Sigma/a\\ r|_{t=0}=r_0,\ p|_{t=0}=p_0,\ \rho|_{t=0}=\rho_0,\ w|_{t=0}=0\end{gathered} \tag{4.2}$$

where ρ_0 is the density of the liquid at pressure p_0 and r_0 the Lagrange coordinate of the liquid particles; the origin is taken at the centre of the bubble and the Lagrange coordinates coincide with the Euler coordinates at the starting time. The equation of state of the liquid (the last equation of (4.1)) will be the equation in the acoustic approximation

$$p=p_0+C^2(\rho-\rho_0) \tag{4.3}$$

The equation of state of the gas is taken as

$$p_g(a)=\left(p_0+\frac{2\Sigma}{a_0}\right)\left(\frac{a_0}{a}\right)^{3\gamma} \tag{4.4}$$

corresponding to adiabatic behaviour of the bubble. The pressure at the boundary R varies sinusoidally

$$p_R(t)=p_0(1-P_R\sin\omega t) \tag{4.5}$$

For convenience in numerical integration, the problem was expressed in dimensionless form (at $\Sigma=0$)

$$V=\frac{\partial \bar r^3}{\partial \bar r_0^3},\ \frac{\partial w}{\partial \tau}=-\left(\frac{\bar r}{\bar r_0}\right)^2\frac{\partial p}{\partial \bar r_0},\ \frac{\partial \bar r}{\partial \tau}=\bar w,\ \bar p=1+\frac{C^2}{p_0\rho_0}\left(\frac{1}{V}-1\right)$$

$$\bar p\left(\frac{R}{a_0},\tau\right)=p_R(\tau),\ \bar p(1,\tau)=\bar r^{-3\gamma}(1,\tau)$$

$$\bar r|_{\tau=0}=\bar r_0,\ \bar p|_{\tau=0}=1,\ \bar w|_{\tau=0}=0,\ V|_{\tau=0}=1$$

where

$$V=\frac{\rho_0}{\rho},\ \bar p=\frac{p}{p_0},\ \bar r=\frac{r}{a_0},\ \bar r_0=\frac{r_0}{a_0},\ \bar w=\frac{w}{C_*},\ \tau=\frac{C_*}{a_0}t\left(C_*^2=\frac{p_0}{\rho_0}\right)$$

are dimensionless variables.

Figures 2 and 3 illustrate the result of computations carried out using a difference scheme [8] (the solid curves) and our system of finite-difference differential equations (2.19) (the dashed curves) for a_0 = 10 μm, R = 5 cm, $p_0 = 10^5$ Pa, $\rho_0 = 10^3$ kg/m^3, C = 1500 m/s, and $\omega=\omega_3=2\pi\times45$ kHz. The comparison was carried out for the time dependence of the bubble radius, with the Mach number monitored from the velocity of the bubble wall. It can be seen from the figures that at small amplitudes of the "flask" pressure (P_R = 0.01, Fig. 3) the Mach numbers are very small ($|M| < 0.001$) and model (2.19) gives almost exact agreement with the solution of the complete system. If the amplitude of the external pressure is increased by a factor of six (P_R = 0.006), the Mach numbers increase by a factor of 50 ($|M| < 0.005$) and the approximate solution deviates from the exact solution.

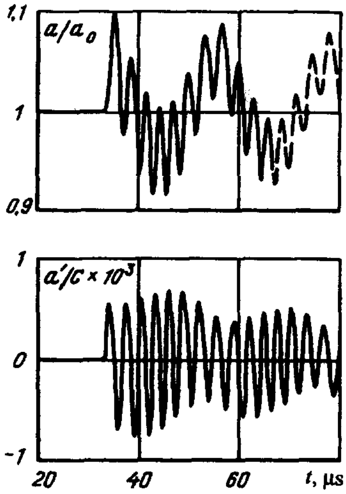

Fig. 2.

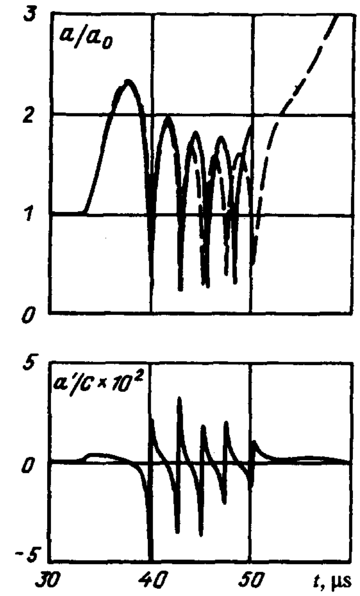

Fig. 3.

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